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Perturbation theory for approximately integrable partial differential equations, and the change of amplitude of solitary-wave solutions of the BBM equation

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In this paper we use existing perturbation theories for partial differential equations that are approximately integrable. These methods are then used to study the BBM equations, considered as a perturbation of the KdV equations. We show, analytically, that the amplitudes of two solitary waves are altered after interaction. The change in amplitude is also calculated.

1. Introduction

The name soliton was first used in connection with the solitary-wave solution of the Korteweg–de Vries equation. This equation was first derived by Korteweg & de Vries (1895) as a model for the unidirectional propagation of water waves of small amplitude and long wavelength. One set of solutions of the KdV equation is a family of solitary waves, and the numerical studies of Zabusky & Kruskal (1965) indicated that the result of the nonlinear interaction of a pair of unequal solitary waves leaves the waves unaltered, except for a phase shift. The proof that this type of interaction happened exactly came from the inverse-scattering method for the solution of partial differential equations. This method of solution can be applied to any partial differential equation that is termed integrable. Such an equation can be thought of as a generalization, to infinite dimensions, of an integrable Hamiltonian system of ordinary differential equations where motion takes place on a finite-dimensional torus. This torus is determined by integrals of the motion. The inverse-scattering method can also be viewed in this light. For partial differential equations the motion takes place on an infinite-dimensional torus, determined by an infinite number of conserved quantities or integrals of the motion.

An alternative model to the KdV equation, first used by Peregrine (1964) for its advantages in numerical computations, is the BBM equation. This equation has been studied extensively by Benjamin, Bona & Mahoney (1972). Recent results by McLeod & Olver (see Bona, Pritchard & Scott 1980) indicate that this equation, and the full Euler equations from which both the KdV and BBM equations are derived, are not integrable, since they have only a finite number of conserved quantities. The term non-integrable is used in a vague sense here, but such equations can be thought of as a generalization, to infinite dimensions, of a non-integrable Hamiltonian system of ordinary differential equations where often there is only one conserved quantity, namely the Hamiltonian. These systems of ordinary differential equations have been studied by many authors and a summary of methods available can be found in Arnol'd (1983).

A recent theorem by Bona (1978) proves that the solutions of the KdV and BBM equations remain close for times that are not too large. However, the bound on the difference grows with time, and numerical evidence indicates that, for the BBM equation, two unequal solitary waves produce a third 'rarefaction' wave or dispersive tail. As a result, the final amplitudes of the two waves are slightly altered after interaction. Details of this numerical investigation are given in Abdulloev, Bogolubsky & Makhankov (1976) and Bona *et al.* (1980).

Hamiltonian systems of ordinary differential equations that are approximately integrable can be solved by perturbation methods; again see, for example, Arnol'd (1983). In this paper we use a similar technique to obtain the solution of the BBM equation from its approximate solution given by the KdV equation. This will predict the change of amplitude of two solitary waves after interaction. In §2 we develop the theory applicable to any partial differential equation that can be approximated by an integrable equation. In §3 we apply this theory to the BBM equation and consider the interaction of two solitary waves.

2. Scattering theory

In this section we review the scattering–inverse scattering theory for integrable partial differential equations, and show how this can be modified for equations that are perturbations of integrable equations. The notation follows closely that of McKean & van Moerbeke (1975) and McKean (1978), and readers are referred to these articles for a fuller description of scattering theory and additional references.

We first consider $u \equiv u(x) \in C^{\infty}_{\downarrow}$, the class of infinitely differentiable functions vanishing rapidly at $\pm\infty$. Let D denote $\mathrm{d}/\mathrm{d}x$ and define a linear operator Q acting on functions of x by

$$\mathrm{Q} = -\mathrm{D}^2 + u. \tag{2.1}$$

Now let f be a solution of

$$\mathrm{Q}(f) = \lambda f^{\dagger}. \tag{2.2}$$

For $0 \leqslant \lambda < \infty$ the spectrum of Q is continuous. If we write $\lambda = k^2$ then with $k \neq 0$ there are two independent solutions which have the following behaviour at ∞:

$$\left.\begin{aligned} \mathring{f}_1 &\sim \mathrm{e}^{\mathrm{i}kx} + S_{12}\,\mathrm{e}^{-\mathrm{i}kx},\ S_{11}\,\mathrm{e}^{\mathrm{i}kx} \\ \mathring{f}_2 &\sim S_{22}\,\mathrm{e}^{-\mathrm{i}kx},\ \mathrm{e}^{-\mathrm{i}kx} + S_{21}\,\mathrm{e}^{\mathrm{i}kx} \end{aligned}\right\} \quad x \to -\infty, +\infty, \tag{2.3}$$

where

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

is unitary with $S_{11} = S_{22}$. Then the Wronskian

$$W(\mathring{f}_1, \mathring{f}_2) = \mathring{f}_1\mathring{f}_2' - \mathring{f}_1'\mathring{f}_2 = \text{const} = -2\mathrm{i}S_{11}\,k. \tag{2.4}$$

In addition to the continuous spectrum there may be a finite number n of additional discrete eigenvalues $\lambda = -k_i^2, i = 1, \ldots, n$. Now there is only one eigenfunction $\mathring{f} \sim 1 \times \mathrm{e}^{-k_i x}$ as $x \to +\infty$ (and $\sim$ constant $\times\, \mathrm{e}^{+k_i x}$ as $x \to -\infty$). We also define the norming constant c_i by the equation

$$c_i^2 \int_{-\infty}^{\infty} |\mathring{f}|^2\,\mathrm{d}k = 1. \tag{2.5}$$

† We use the notation $\mathrm{Q}(v) \equiv -v'' + uv$ for Q acting on the function v and reserve the notation $\mathrm{Q}v$ for the linear operator of multiplication by v followed by Q. Thus $\mathrm{Q}v(w)$ is the operator $\mathrm{Q}v$ acting on w and is equal to $-(vw)'' + uvw$ and is of course in this instance $\mathrm{Q}(vw)$.

The spectral data S_{21} and the numbers $\{k_i, c_i\}$, $i = 1, \ldots, n$ are called the augmented spectral data, and the fundamental theorem of scattering theory is that the map from u to the augmented spectral data is 1:1 and onto (see, for example, Levinson 1953 and Deift & Trubowitz 1979). The backward or inverse scattering is achieved via the Gelfand–Levitan integral equation (see Gelfand & Levitan 1951 or Agranovitch & Marcenko 1963).

Now suppose that X and K are skew-symmetric operators, acting on functions of x, with the following properties: first, that X induces a local Hamiltonian flow in $C_{\downarrow}^{\infty}$, and second, that the following expression is an identity:

$$\mathrm{X}(u) = [Q, \mathrm{K}], \tag{2.6}$$

where $[Q, \mathrm{K}]$ is the usual commutator $Q\mathrm{K} - \mathrm{K}Q$.

The first property means that the equation

$$\frac{\partial u}{\partial t} = \mathrm{X}(u) \tag{2.7}$$

can be solved in $C_{\downarrow}^{\infty}$ for all time $-\infty < t < \infty$. (See Arnol'd 1983 for more general information on Hamiltonian systems.)

We shall work as far as possible with the formal operators X and K, but with $\mathrm{X} \equiv -\frac{1}{2}\mathrm{D}^3 + u\mathrm{D} + \mathrm{D}u$, the equation

$$\frac{\partial u}{\partial t} = \mathrm{X}(u) = -\tfrac{1}{2}u''' + uu' + (u^2)' = 3uu' - \tfrac{1}{2}u''',$$

where

$$' \equiv \frac{\partial}{\partial x}, \tag{2.8}$$

is the Korteweg–de Vries equation, and Lax (1968) discovered the identity

$$\frac{\partial u}{\partial t} \equiv \frac{\partial Q}{\partial t} = [Q, \mathrm{K}], \tag{2.9}$$

where $\mathrm{K} \equiv 2\mathrm{D}^3 - \frac{3}{2}(u\mathrm{D} + \mathrm{D}u)$.

The second property means that under (2.7), $-\mathrm{K}$ is the infinitesimal transformation, and the corresponding eigenfunctions move according to the equation

$$\frac{\partial f}{\partial t} = -\mathrm{K}(f). \tag{2.10}$$

For the KdV equation, the scattering method yields the results

$$S_{11}(t) = S_{11}(t_0), \tag{2.11}$$

$$S_{12}(t) = S_{12}(t_0)\exp[-4\mathrm{i}k^3(t-t_0)], \tag{2.12}$$

$$k_i = k_i(t_0), \quad i = 1, \ldots, n, \tag{2.13}$$

$$c_i^2 = c_i^2(t_0)\exp[4k_i^3(t-t_0)], \quad i = 1, \ldots, n, \tag{2.14}$$

where the constants $S_{11}(t_0)$, $S_{12}(t_0)$, $k_i(t_0)$ and $c_i(t_0)$ are determined from the initial solution u at $t = t_0$.

If now we suppose that u satisfies the perturbed equation

$$\frac{\partial u}{\partial t} = \mathrm{X}(u) + \epsilon\mathrm{X}_1(u) \tag{2.15}$$

then the methods of Karpman & Maslov (1977*a*, *b*, *c*), Keener & McLaughlin (1977), and Kaup & Newell (1978) allow us to obtain the results

$$S_{11}(t) = S_{11}(t_0)+\frac{\epsilon}{2ik}\int_{t_0}^{t} I_{12}(\tau)\,\mathrm{d}\tau, \tag{2.16}$$

$$S_{12}(t) = S_{12}(t_0)\exp[-4ik^3(t-t_0)]+\frac{\epsilon}{2ik}\int_{t_0}^{t}\exp[4ik^3(\tau-t)]\,I_{11}(\tau)\,\mathrm{d}\tau, \tag{2.17}$$

$$k_i^2(t) = k_0^2(t_0)-\epsilon\int_{t_0}^{t} c_i^2(\tau)\,I_1(\tau)\,\mathrm{d}\tau, \tag{2.18}$$

$$c_i^2(t) = c_i^2(t)-\epsilon\int_{t_0}^{t} c_i^2(\tau)\exp[-4k_i^3(\tau-t)]\,I_2(\tau)\,\mathrm{d}\tau, \tag{2.19}$$

where

$$I_{ij}(t) = \int_{-\infty}^{\infty}\mathring{f}_i\mathring{f}_j\,\mathrm{X}_1(u)\,\mathrm{d}x, \tag{2.20}$$

$$I_1(t) = \int_{-\infty}^{\infty}\mathring{f}^2\mathrm{X}_1(u)\,\mathrm{d}x, \tag{2.21}$$

and

$$I_2(\tau) = 2\int_{-\infty}^{\infty}\mathring{f}^2\int_{x}^{\infty}\mathring{f}^{-2}\int_{x}^{\infty}\{\mathrm{X}_1(u)-c_i^2 I_1)\mathring{f}^2-\mathring{f}\mathring{f}'c_i^2 I_1/k_i\}\,\mathrm{d}x+\frac{c_i^2}{k_i}I_1\int_{-\infty}^{\infty}x\mathring{f}^2\,\mathrm{d}x. \tag{2.22}$$

Equations (2.16)–(2.19) are simply a restatement of the original partial differential equation (2.8), and should be regarded as integral equations, since the integrals defined by (2.20)–(2.22) are functionals of S_{11}, S_{12}, k_i and c_i via their dependence on u. However if we expand u as a power series in ϵ with the first approximation, u_0, satisfying (2.8), then u_0 determines the scattering data correct to order ϵ, and the inverse-scattering method can be used to determine u correct to order ϵ.

3. Application to BBM equation

3.1. *The BBM equation*

The BBM equation applicable to water waves of small amplitude and long wavelength is

$$\frac{\partial V}{\partial t^*}+\frac{\partial V}{\partial x^*} = \epsilon\left(3V\frac{\partial V}{\partial x^*}+\frac{1}{2}\frac{\partial^3 V}{\partial x^{*2}\,\partial t^*}\right). \tag{3.1}$$

In the context of the water-wave problem, $(-V)$ represents the vertical displacement of the surface of the liquid from its equilibrium position, scaled so that the maximum displacement is of order 1. x^*, the horizontal coordinate and t^*, the time, are both dimensionless independent variables, with length- and timescales being chosen so that all derivatives are of order 1. For a fuller description of the scaling and a derivation of the more familiar KdV equation see, for example, Miles (1977). For a fuller description of the BBM equation see, for example, Benjamin *et al.* (1972).

If we introduce a moving coordinate x and a scaled time coordinate t given by

$$x = x^*-t^*, \quad t = \epsilon t^*, \tag{3.2}$$

and write

$$V(x^*, t^*) = u(x,t), \tag{3.3}$$

then (3.1) becomes

$$\frac{\partial u}{\partial t} = 3u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} + \tfrac{1}{2}\epsilon\frac{\partial^3 u}{\partial x^2\,\partial t} = \mathrm{X}(u) + \tfrac{1}{2}\epsilon \mathrm{D}^2(u_t). \tag{3.4}$$

To write this in Hamiltonian form we simply invert the operator $I - \frac{1}{2}\epsilon \mathrm{D}^2$ to obtain

$$\frac{\partial u}{\partial t} = \frac{1}{(2\epsilon)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right]\mathrm{X}(u(\xi))\,\mathrm{d}\xi. \tag{3.5}$$

However for most purposes it will be sufficient to use the approximation

$$\frac{\partial u}{\partial t} = \mathrm{X}(u) + \tfrac{1}{2}\epsilon \mathrm{D}^2(\mathrm{X}(u)). \tag{3.6}$$

3.2. *The solitary-wave solution of the KdV equation*

The multisoliton solution of the KdV equation is best obtained via the transformations used by Whitham (1974). He introduces variables P and E defined by

$$u = \frac{\partial P}{\partial x}, \quad P = -2\mathrm{D}(\log E), \tag{3.7}$$

so that

$$\frac{\partial P}{\partial t} = \frac{3}{2}\left(\frac{\partial P}{\partial x}\right)^2 - \frac{\partial^3 P}{\partial x^3} \tag{3.8}$$

and

$$E\left(2\frac{\partial E}{\partial t} + E'''\right)' - E'\left(2\frac{\partial E}{\partial t} + E'''\right) + 3(E''^2 - E'E''') = 0, \tag{3.9}$$

where

$$' \equiv \frac{\partial}{\partial x}.$$

The solution corresponding to one solitary wave is

$$E = 1 + \exp\{-(\theta - \theta_0)\} \tag{3.10}$$

and

$$u = -\tfrac{1}{2}k^2\,\mathrm{sech}^2\,\tfrac{1}{2}(\theta - \theta_0), \tag{3.11}$$

where k and θ_0 are constants and $\theta = kx - \frac{1}{2}k^3 t$.

If we have two solitary waves, then the corresponding solution is

$$E = 1 + E_1 + E_2 + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}E_1 E_2 \tag{3.12}$$

and

$$u = -2\,\frac{k_1^2 E_1 + k_2^2 E_2 + 2(k_1-k_2)^2 E_1 E_2 + \dfrac{(k_1-k_2)^2}{(k_1+k_2)^2}E_1 E_2(k_2^2 E_1 + k_1^2 E_2)}{E^2}, \tag{3.13}$$

where

$$E_1 = \exp\{-(\theta_1 - \theta_0)\}, \quad \theta_1 = k_1 x - \tfrac{1}{2}k^3 t. \tag{3.14}$$

We then have the following results for the eigenfunctions. F_1 and F_2 satisfy

$$-F_i'' + uF_i = -\tfrac{1}{4}k_i^2 F_i \tag{3.15}$$

and are given by

$$F_i = \frac{E_i^{\frac{1}{2}}\left(1 + \dfrac{k_i - k_j}{k_i + k_j}E_j\right)}{E} \quad (i = 1, 2, \quad j \neq i). \tag{3.16}$$

Then
$$\int_{-\infty}^{\infty} F_i^2 = \frac{1}{k_i} \tag{3.17}$$

and
$$u = -2\Sigma k_i^2 F_i^2 \equiv \frac{2(EE'' - E'^2)}{E^2}. \tag{3.18}$$

(Note that the discrete eigenvalues are $\lambda = -(\frac{1}{2}k_1)^2$ and $-(\frac{1}{2}k_2)^2$. The factor of $\frac{1}{2}$ has been introduced for convenience.)

3.3. *The two-solitary-wave solution of BBM*

To obtain an approximation to the two-solitary-wave solutions of BBM, we use (3.13) as the initial condition as $t \to -\infty$. Then (3.13) represents the first approximation to u, and (2.11)–(2.14) represent the first approximation to the scattering data. Equations (3.13)–(3.16) are then sufficient to determine k_i^2 correct to order ϵ. From (3.6) we have $\mathrm{X}_1(u) = \frac{1}{2}\mathrm{D}^2(\mathrm{X}(u))$, and correct to order ϵ (2.18) yields

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}(k_1^2)\int_{-\infty}^{\infty} F_1^2 = -\tfrac{1}{2}\epsilon \int_{-\infty}^{\infty} F_1^2\,\mathrm{D}^2(\mathrm{X}(u))$$
$$= \epsilon \int_{-\infty}^{\infty} \{k_1^2 F_1^2\,\mathrm{D}^2(\mathrm{X}(F_1^2)) + k_2^2 F_1^2\,\mathrm{D}^2(\mathrm{X}(F_2^2))\}. \tag{3.19}$$

Then using (A 2) and (3.17) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}k_1 = -\epsilon \int_{-\infty}^{\infty} \{k_1^4 F_1^2\,\mathrm{D}^3(F_1^2) + k_2^4 F_1^2\,\mathrm{D}^3(F_2^2)\}$$
$$= -\epsilon k_2^4 \int_{-\infty}^{\infty} F_1^2\,\mathrm{D}^3(F_2^2), \tag{3.20}$$

since D^3 is antisymmetric and $F_i \in C_\downarrow^\infty$. Therefore

$$\frac{\mathrm{d}k_1}{\mathrm{d}t} = -2\epsilon k_2^4 \int_{-\infty}^{\infty} F_1^2(u\mathrm{D} + \mathrm{D}u - \mathrm{X})(F_2^2)$$
$$= -2\epsilon k_2^4 \int_{-\infty}^{\infty} \{2F_1^2 F_2^{2\prime} u + F_1^2 F_2^2 u' - F_1^2\,\mathrm{X}(F_2^2)\}$$
$$= -2\epsilon k_2^4 \int_{-\infty}^{\infty} \{(F_1^2 F_2^{2\prime} - F_1^{2\prime} F_2^2)\,u - F_1^2\,\mathrm{X}(F_2^2)\}$$
$$= -2\epsilon k_2^4 \int_{-\infty}^{\infty} (F_1^2 F_2^{2\prime} - F_1^{2\prime} F_2^2)\,u, \tag{3.21}$$

since
$$\int_{-\infty}^{\infty} k_2^2 F_1^2\,\mathrm{X}(F_2^2) = \int_{-\infty}^{\infty} F_1^2(k_1^2\,\mathrm{X}(F_1^2) + k_2^2\,\mathrm{X}(F_2^2))$$
$$= -\int_{-\infty}^{\infty} \tfrac{1}{2}F_1^2\,\mathrm{X}(u)$$
$$= 0; \tag{3.22}$$

see Appendix A.

By using (3.16) and (3.18) we obtain, after considerable algebra (see Appendix A),

$$\frac{dk_1}{dt} = -\tfrac{2}{3}\epsilon k_2^4\left(\frac{k_1-k_2}{k_1+k_2}\right)I, \tag{3.23}$$

where
$$I = k_1 k_2(k_1-k_2)\int_{-\infty}^{\infty}\frac{(E_2-E_1)\,E_1E_2}{E^3}\,dx. \tag{3.24}$$

Similarly we obtain

$$\frac{dk_2}{dt} = +\tfrac{2}{3}\epsilon k_1^4\left(\frac{k_1-k_2}{k_1+k_2}\right)I. \tag{3.25}$$

As a consequence of (3.23) and (3.25) we obtain a conservation law

$$k_1^5+k_2^5 = \text{const}, \tag{3.26}$$

although this applies to the approximation (3.6), and not to the full BBM (3.5).

We now pose the following problem. If we start with two given solitary waves at $t=-\infty$ what will be the form of solution as $t\to+\infty$?

In order to compute the solution u correct to order ϵ we need to find the scattering data correct to order ϵ and then use the inverse scattering method. However if we are only interested in the emerging solitary waves, then (3.23) and (3.25) are sufficient to determine their amplitudes and the phase change may be computed by solving (2.19) for c_i.

Since both k_1 and k_2 are slowly varying functions of time, we might think that we could work out the total change as

$$[k_1] = \int_{-\infty}^{\infty}\frac{dk_1}{dt}\,dt = -\tfrac{2}{3}k_2^4\left(\frac{k_1-k_2}{k_1+k_2}\right)\int_{-\infty}^{\infty} I\,dt, \tag{3.27}$$

regarding k_1 and k_2 as constant. However, if we introduce the integral

$$J(k_1,k_2,t) = \int_{-\infty}^{\infty}\frac{E_1E_2}{E^2}\,dx, \tag{3.28}$$

we may show (see Appendix B) that

$$\frac{\partial J}{\partial t} = I, \tag{3.29}$$

the partial derivative denoting the derivative with respect to t keeping k_1 and k_2 constant. Equation (3.27) would then give as solution

$$[k_1] = 0. \tag{3.30}$$

Thus, to find the total change in k_1 we need to proceed to the second approximation. There are two terms that arise in the second approximation. One comes from the fact that we have approximated the full equation (3.4) by (3.6), and a second term comes from the expansion of (3.23) and (3.25) to second order.

We shall show (see Appendix C) that (3.30) also holds for the expression for dk_1/dt derived from the full equation (3.4), so that we only need to concentrate on the second term.

With $L_1 = \tfrac{2}{3}k_2^4(k_1-k_2)/(k_1+k_2)$ and L_2 defined similarly, we write

$$\frac{dk_1}{dt} = -\epsilon L_1\frac{\partial J}{\partial t},\quad \frac{dk_2}{dt} = \epsilon L_2\frac{\partial J}{\partial t}. \tag{3.31}$$

Therefore

$$\frac{dk_1}{dt} = -\epsilon\frac{d}{dt}(L_1 J) + \epsilon\left(\frac{\partial L_1}{\partial k_2}\frac{dk_2}{dt} + \frac{\partial L_1}{\partial k_1}\frac{dk_1}{dt}\right)J + \epsilon L_1\left(\frac{\partial J}{\partial k_1}\frac{dk_1}{dt} + \frac{\partial J}{\partial k_2}\frac{dk_2}{dt}\right)$$

$$= -\epsilon\frac{d}{dt}(L_1 J) - \epsilon^2\left(L_1\frac{\partial L_1}{\partial k_1} - L_2\frac{\partial L_1}{\partial k_2}\right)J\frac{\partial J}{\partial t} - \epsilon^2 L_1^2\left(\frac{\partial J}{\partial k_1} - \left(\frac{k_1}{k_2}\right)^4\frac{\partial J}{\partial k_2}\right)\frac{\partial J}{\partial t}, \quad (3.32)$$

so that

$$\int_{-\infty}^{\infty}\frac{dk_1}{dt}dt = [-\epsilon L_1 J]_{-\infty}^{\infty} - \epsilon^2\left(L_1\frac{\partial L_1}{\partial k_1} - L_2\frac{\partial L_1}{\partial k_2}\right)[\tfrac{1}{2}J^2]_{-\infty}^{\infty}$$
$$-\epsilon^2 L_1^2\int_{-\infty}^{\infty}\left(\frac{\partial J}{\partial k_1} - \left(\frac{k_1}{k_2}\right)^4\frac{\partial J}{\partial k_2}\right)\frac{\partial J}{\partial t}dt, \quad (3.33)$$

where in the ϵ^2 terms we have assumed that the variation of k_1 and k_2 with time can be ignored. Thus,

$$[k_1] = -\tfrac{4}{9}\epsilon^2 k_2^4\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\int_{-\infty}^{\infty}\left(k_2^4\frac{\partial J}{\partial k_1} - k_1^4\frac{\partial J}{\partial k_2}\right)\frac{\partial J}{\partial t}dt. \quad (3.34)$$

After some considerable algebra (see Appendix D) this reduces to

$$[k_1] = \epsilon^2 k_2^5 G(\beta)\, I_1(\beta), \quad (3.35)$$

where
$$G(\beta) = -\frac{1}{270}\frac{1+10\beta^2+5\beta^4+2(7+10\beta^2-\beta^4)\beta^2\log\beta/(1-\beta^2)}{1+\beta}, \quad (3.36)$$

$$I_1 = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{30^{\frac{1}{2}}\sinh(y-\tau)}{\{\beta\cosh(y/\beta)+\cosh(y-\tau)\}^3}dy\right\}^2 d\tau, \quad (3.37)$$

and
$$\beta = \frac{k_2-k_1}{k_1+k_2}. \quad (3.38)$$

The integral I_1 has been normalized so that $I_1(1)$ equals one. Then using the result

$$\int_{-\infty}^{\infty}\frac{dx}{\cosh x+\cosh\gamma} = 2\gamma\operatorname{cosech}\gamma \sim 4\gamma\, e^{-\gamma} \quad \text{as } \gamma\to\infty, \quad (3.39)$$

we may obtain the result

$$I_1(\beta) \sim 32(\beta\log\beta)^2 \quad \text{as } \beta\to 0.$$

For values of β in the range $0 < \beta < 1$ the integral is easily computed, and is shown in figure 1.

The multiplier G has the following properties:

$$G(\beta) \sim \tfrac{4}{270}(1-\beta), \quad \beta\to 1,$$
$$G(\beta) > 0 \quad 0.318\ldots < \beta < 1,$$
$$G(\beta) < 0 \quad 0 \leqslant \beta < 0.318\ldots,$$

and
$$G(0) = -\tfrac{1}{270}.$$

Thus the change in k_1 is positive for $\beta > 0.318\ldots$ and negative for $\beta < 0.318$. The graph of the change in k_1 appears in figure 2. The factor $\epsilon^2 k_2^5$ has been omitted and the scale on the vertical axis has been multiplied by 10^3 for convenience.

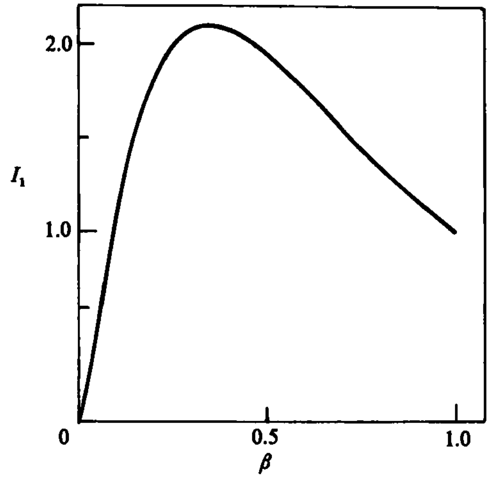

FIGURE 1. Graph of the integral I_1 defined by (3.37) as a function of β in the range $0 \leqslant \beta \leqslant 1$.

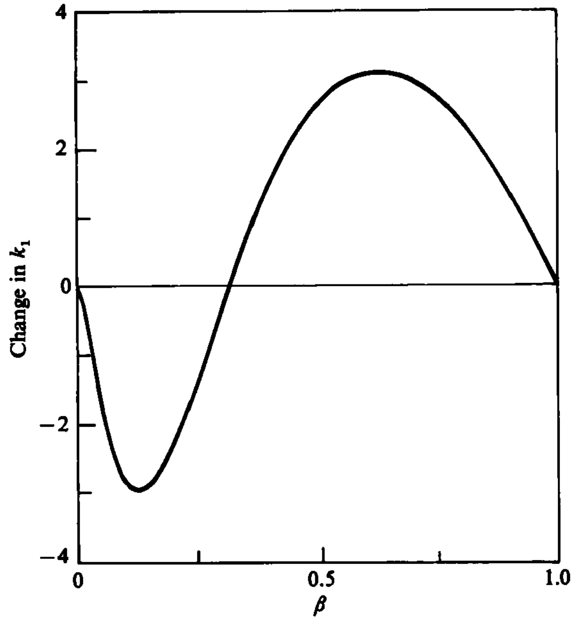

FIGURE 2. Graph of the change in k_1 as a function of β in the range $0 \leqslant \beta \leqslant 1$.

4. Discussion of the results

The evolution (3.23) and (3.25) shows that the rate of change of k_1 and k_2 is of order ϵ. Since from (3.28) J is never negative, and tends to zero as $t \rightarrow \infty$, the sign of $\partial J/\partial t$ is positive for large negative times. If k_2 is greater than k_1, (3.23) then gives the following result:

$$\frac{dk_1}{dt} > 0 \quad \text{for} \quad t \text{ large and negative.} \tag{4.1}$$

However (3.30) implies that

$$[k_1] = 0 \quad \text{to order } \epsilon. \tag{4.2}$$

This means that k_1 starts to increase and reaches a maximum before decreasing to its final value given by (3.35). This value may be positive or negative, depending on the ratio of k_1 to k_2. However, it is always of order ϵ^2. This has important consequences when comparing with numerical results. To get good comparison, the two waves must have an initial separation which ensures that the integral I defined by (3.24) is $o(\epsilon)$.

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Appendix A

In this Appendix we prove

$$\int_{-\infty}^{\infty} f^2 \mathrm{X}(u) = 0$$

where f is the eigenfunction satisfying (2.2), with λ belonging to the discrete spectrum.

The skew symmetry of X gives the following identity for any functions α and β:

$$\alpha \mathrm{X}(\beta) + \beta \mathrm{X}(\alpha) = \mathrm{D}\{2\alpha\beta u - \tfrac{1}{2}(\alpha\beta'' - \alpha'\beta' + \alpha''\beta)\}, \tag{A 1}$$

so that with $\alpha = f^2$ and $\beta = 1$ and using the result (see McKean & van Moerbeke)

$$\mathrm{X}(f^2) = 2\lambda f^{2\prime}, \tag{A 2}$$

we obtain

$$f^2 \mathrm{X}(1) = \mathrm{D}\{-2\lambda f^2 + 2f^2 u - \tfrac{1}{2} f^{2\prime\prime}\}. \tag{A 3}$$

Thus

$$\int_{-\infty}^{\infty} f^2 \mathrm{X}(1) = 0 \tag{A 4}$$

since $u \to 0$ and $f \to 0$ at ∞.

The skew symmetry of X also gives

$$\int_{-\infty}^{\infty} f^2 \mathrm{X}(u) = -\int_{-\infty}^{\infty} u \mathrm{X}(f^2), \tag{A 5}$$

since $u \to 0$ at ∞. Therefore

$$\int_{-\infty}^{\infty} f^2 \mathrm{X}(u) = -2\lambda \int_{-\infty}^{\infty} u f^{2\prime} = +2\lambda \int_{-\infty}^{\infty} f^2 \mathrm{D}(u) = 2\lambda \int_{-\infty}^{\infty} f^2 \mathrm{X}(1), \tag{A 6}$$

so that

$$\int_{\infty}^{\infty} f^2 \mathrm{X}(u) = 0. \tag{A 7}$$

Appendix B

We prove the formulae (3.24) and (3.29). From the definition of the eigenfunctions (3.16) we obtain

$$F_1^2 F_2^2 \mathrm{D}\left\{\log\frac{F_2^2}{F_1^2}\right\} = F_1^2 F_2^2\left\{k_1-k_2+2\frac{k_1-k_2}{k_1+k_2}k_2 E_2\Big/\left(1+\frac{k_1-k_2}{k_1+k_2}E_2\right)\right.$$

$$\left.+2\frac{k_1-k_2}{k_1+k_2}k_1 E_1\Big/\left(1+\frac{k_1-k_2}{k_1+k_2}E_1\right)\right\}$$

$$= (k_1-k_2)F_1^2 F_2^2\left\{1+\frac{2}{k_1+k_2}\right.$$

$$\left.\times\left\{\frac{k_1 E_1+k_2 E_2+(k_1-k_2)^2/(k_1+k_2)E_1 E_2}{1+(k_1-k_2)/(k_1+k_2)(E_2-E_1)-(k_1-k_2)^2/(k_1+k_2)^2 E_1 E_2}\right\}\right\}$$

$$= (k_1-k_2)F_1^2 F_2^2\left\{1-\frac{2}{k_1+k_2}\left(\frac{E'}{E+2E'/(k_1+k_2)}\right)\right\}$$

$$= (k_1-k_2)F_1^2 F_2^2\frac{(k_1+k_2)E}{((k_1+k_2)E+2E')}. \tag{B 1}$$

Also we obtain

$$F_1 F_2 = \frac{(E_1 E_2)^{\frac{1}{2}}}{E^2}\left\{1+\frac{k_1-k_2}{k_1+k_2}(E_2-E_1)-\frac{(k_1-k_2)^2}{(k_1+k_2)^2}E_1 E_2\right\}$$

$$= \frac{(E_1 E_2)^{\frac{1}{2}}}{E^2}\left\{E+\frac{2E'}{k_1+k_2}\right\}, \tag{B 2}$$

so that

$$F_1^2 F_2^{2\prime}-F_1^{2\prime}F_2^2 = \left(\frac{k_1-k_2}{k_1+k_2}\right)\frac{E_1 E_2}{E^3}\{2E'+(k_1+k_2)E\}. \tag{B 3}$$

Thus using (3.18) we obtain

$$\int_{-\infty}^{\infty}(F_1^2 F_2^{2\prime}-F_1^{2\prime}F_2^2)u = -2\frac{k_1-k_2}{k_1+k_2}\int_{-\infty}^{\infty}\frac{E_1 E_2}{E^5}\{2E'+(k_1+k_2)E\}(EE''-E'^2). \tag{B 4}$$

The term $E'^3E_1 E_2/E^5$ may be integrated as

$$\int_{-\infty}^{\infty}\frac{E'^3E_1 E_2}{E^5} = \left[-\frac{\frac{1}{4}E'^2E_1 E_2}{E^4}\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\left\{\frac{\frac{1}{2}E'E''E_1 E_2}{E^4}-\frac{\frac{1}{4}(k_1+k_2)E'^2E_1 E_2}{E^4}\right\}$$

$$= \frac{1}{4}\int_{-\infty}^{\infty}\frac{E'\{2E''-(k_1+k_2)E'\}E_1 E_2}{E^4}. \tag{B 5}$$

Repeated integration by parts gives the result

$$\int_{-\infty}^{\infty}(F_1^2 F_2^{2\prime}-F_1^{2\prime}F_2^2)u = -\frac{2}{3}\frac{k_1-k_2}{k_1+k_2}\int_{-\infty}^{\infty}\frac{E_1 E_2}{E^3}(E'''+\tfrac{3}{2}(k_1+k_2)E''+\tfrac{1}{2}(k_1+k_2)^2E')$$

$$= \frac{k_1-k_2}{3(k_1+k_2)}\int_{-\infty}^{\infty}\frac{k_1 k_2(k_1-k_2)(E_2-E_1)E_1 E_2}{E^3}, \tag{B 6}$$

or

$$I = k_1 k_2(k_1-k_2)\int_{-\infty}^{\infty}\frac{(E_2-E_1)E_1 E_2}{E^3}, \tag{B 7}$$

which is (3.24). From (3.28) we obtain

$$\frac{\partial J}{\partial t} = \int_{-\infty}^{\infty} \left\{\frac{E_1 E_2}{2E^2}(k_1^3+k_2^3) - \frac{E_1 E_2}{E^3}\left(k_1^3 E_1 + k_2^3 E_2 + \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 (k_1^3+k_2^3) E_1 E_2\right)\right\}$$

$$= \int_{-\infty}^{\infty} \frac{E_1 E_2}{2E^2}(k_1^3+k_2^3) - \frac{E_1 E_2}{E^3}\left\{k_1^3 E_1 + k_2^3 E_2 - \frac{k_1^3+k_2^3}{k_1+k_2}(E' + k_1 E_1 + k_2 E_2)\right\}$$

$$= \int_{-\infty}^{\infty} \left\{\frac{E_1 E_2}{2E^3}\left(E + \frac{2E'}{k_1+k_2}\right)(k_1^3+k_2^3) + E_1 E_2 \frac{E_2-E_1}{E^3} k_1 k_2 (k_1-k_2)\right\}$$

$$= k_1 k_2 (k_1-k_2) \int_{-\infty}^{\infty} \frac{E_2-E_1}{E^3} E_1 E_2 + \left[-\frac{E_1 E_2}{2E^2}\frac{k_1^3+k_2^3}{k_1+k_2}\right]_{-\infty}^{\infty}$$

$$= I, \tag{B 8}$$

which is (3.29).

Appendix C

Here we prove that the expression for $\mathrm{d}k_1/\mathrm{d}t$ satisfies (3.30) if u satisfies (3.5).

We first notice from (A 6) that

$$\int_{-\infty}^{\infty} F_1^2 \mathrm{X}(u)\,\mathrm{d}x = 0. \tag{C 1}$$

Thus if u satisfies (3.5) we may write

$$\frac{\mathrm{d}k_1}{\mathrm{d}t} = -2\frac{1}{(2\epsilon)^{\frac{1}{2}}}\int_{-\infty}^{\infty} F_1^2(x) \int_{-\infty}^{\infty} \exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right]\mathrm{X}(u(\xi))\,\mathrm{d}\xi\,\mathrm{d}x$$

$$= -\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1^2(x)\exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right](-2k_1^2\,\mathrm{X}(F_1^2(\xi)) - 2k_2^2\,\mathrm{X}(F_2^2(\xi)))\,\mathrm{d}\xi\,\mathrm{d}x$$

$$= 2k_2^2\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1^2(x)\exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right]\mathrm{X}(F_2^2(\xi))\,\mathrm{d}\xi\,\mathrm{d}x, \tag{C 2}$$

since the kernel is clearly an antisymmetric operator so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1^2(x)\exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right]\mathrm{X}(F_1^2(\xi))\,\mathrm{d}\xi\,\mathrm{d}x \equiv 0. \tag{C 3}$$

(Note here we have not mentioned explicitly the dependence of F_1 on t. In reality $F_1(x) \equiv F_1(x,t)$.) Also, since X is an antisymmetric operator, we may write

$$2k_2^2\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left[-\left(\frac{2}{\epsilon}\right)^{\frac{1}{2}}|x-\xi|\right]\mathrm{X}(F_2^2(\xi))\,\mathrm{d}\xi = \int_{-\infty}^{\infty} F_2^2(\xi)\,\mathrm{K}(x-\xi)\,\mathrm{d}\xi, \tag{C 4}$$

where the kernel K is antisymmetric in the sense that

$$\mathrm{K}(x) = -\mathrm{K}(-x). \tag{C 5}$$

Hence we may write

$$\frac{\mathrm{d}k_1}{\mathrm{d}t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1^2(x)\,F_2^2(\xi)\,\mathrm{K}(x-\xi)\,\mathrm{d}\xi\,\mathrm{d}x, \tag{C 6}$$

so that

$$I = [k_1] = \int_{-\infty}^{\infty}\frac{\mathrm{d}k_1}{\mathrm{d}t}\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1^2(x)\,F_2^2(\xi)\,\mathrm{K}(x-\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t. \tag{C 7}$$

Using the definition of F_1 (3.16) we may obtain

$$F_1^2 = \frac{\cosh^2(\frac{1}{2}k_2 x - \frac{1}{4}k_2^3 t + \frac{1}{2}\theta)\,\mathrm{e}^{-\theta}}{2\{\mathrm{e}^{-\theta}\cosh(\frac{1}{2}(k_1+k_2)\,x - \frac{1}{4}(k_1^3+k_2^3)\,t + \theta) + \cosh(\frac{1}{2}(k_2-k_1)\,x - \frac{1}{4}(k_2^3-k_1^3)\,t)\}^2}, \tag{C 8}$$

where

$$\mathrm{e}^{-\theta} = \left|\frac{k_1-k_2}{k_1+k_2}\right|. \tag{C 9}$$

Then if we make the substitutions

$$\left.\begin{aligned} y &= x - \tfrac{1}{2}(k_1^2 - k_1 k_2 + k_2^2)\,t, \\ \eta &= \xi - \tfrac{1}{2}(k_1^2 - k_1 k_2 + k_2^2)\,t, \\ \tau &= \tfrac{1}{2}k_1 k_2(k_1-k_2)\,t, \end{aligned}\right\} \tag{C 10}$$

in the integral of (C 7), we obtain

$$I = \frac{\mathrm{e}^{-2\theta}}{2k_1 k_2(k_1-k_2)} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\cosh^2(\frac{1}{2}k_2 y + \frac{1}{2}(\theta+\tau))\sinh^2(\frac{1}{2}k_1\eta + \frac{1}{2}(\theta-\tau))\,\mathrm{K}(y-\eta)\,\mathrm{d}\eta\,\mathrm{d}y\,\mathrm{d}\tau}{G(y,\tau)\,G(\eta,\tau)}, \tag{C 11}$$

where

$$G(y,\tau) = \{\mathrm{e}^{-\theta}\cosh^2(\tfrac{1}{2}(k_1+k_2)\,y + \theta) + \cosh(\tfrac{1}{2}(k_2-k_1)\,y + \tau)\}^2. \tag{C 12}$$

If we now make the change of variables

$$y = \omega + v, \quad \eta = v - \omega \tag{C 13}$$

(noting that the precise form of (C 8) and the subsequent expressions alter, depending on whether $k_1 - k_2$ is positive or negative) the integral becomes

$$I = \frac{\mathrm{e}^{-2\theta}}{4k_1 k_2(k_1-k_2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cosh^2(\tfrac{1}{2}k_2(\omega+v) + \tfrac{1}{2}(\theta+\tau)) \times \frac{\sinh^2(\frac{1}{2}k_1(v-\omega) + \frac{1}{2}(\theta-\tau))}{F(\omega+v,\tau)\,G(v-\omega,\tau)}\mathrm{K}(2\omega)\,\mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}\tau. \tag{C 14}$$

Now using (C 5) we may write

$$\int_{-\infty}^{\infty} g(\omega)\,\mathrm{K}(2\omega)\,\mathrm{d}\omega = \int_0^{\infty}(g(\omega) - g(-\omega))\,\mathrm{K}(2\omega)\,\mathrm{d}\omega. \tag{C 15}$$

The numerator of the integrand then becomes, after simplification,

$$\left\{\sinh(\tfrac{1}{2}(k_1+k_2)\,v+\theta)\cosh\tfrac{1}{2}(k_2-k_1)\,\omega + \sinh(\tfrac{1}{2}(k_2-k_1)\,v+\tau)\cosh\frac{(k_1+k_2)}{2}\omega\right\} \times\{\cosh(\tfrac{1}{2}(k_1+k_2)\,v+\theta)\sinh\tfrac{1}{2}(k_2-k_1)\,\omega + \cosh(\tfrac{1}{2}(k_2-k_1)\,v+\tau)\sinh\tfrac{1}{2}(k_1+k_2)\,\omega\}, \tag{C 16}$$

while the denominator becomes

$$\{\mathrm{e}^{-\theta}\cosh(\tfrac{1}{2}(k_1+k_2)\,(v+\omega)+\theta) + \cosh(\tfrac{1}{2}(k_2-k_1)\,(\omega+v)+\tau)^2 \times\{\mathrm{e}^{-\theta}\cosh(\tfrac{1}{2}(k_1+k_2)\,(v-\omega)+\theta) + \cosh(\tfrac{1}{2}(k_2-k_1)\,(v-\omega)+\tau)\}^2. \tag{C 17}$$

If we introduce the change of variables

$$T = \tau - \tfrac{1}{2}(k_2-k_1)\,v, \quad \zeta = v - \frac{2}{k_1+k_2}\theta, \tag{C 18}$$

we may write

$$I = \frac{e^{-2\theta}}{4k_1 k_2(k_1-k_2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{h_1(\omega, \zeta, T)\, \mathrm{K}(2\omega)}{h_2(\omega, \zeta, T)} \mathrm{d}\omega\, \mathrm{d}\zeta\, \mathrm{d}T, \tag{C 19}$$

where

$$h_1 = \{\sinh \tfrac{1}{2}(k_1+k_2)\, \zeta \cosh \tfrac{1}{2}(k_2-k_1)\, \omega + \sinh T \cosh \tfrac{1}{2}(k_1+k_2)\, \omega\}$$
$$\times \{\cosh \tfrac{1}{2}(k_1+k_2)\, \zeta \sinh \tfrac{1}{2}(k_2-k_1)\, \omega + \cosh T \sinh \tfrac{1}{2}(k_1+k_2)\, \omega\} \tag{C 20}$$

and

$$h_2 = \{e^{-\theta} \cosh \tfrac{1}{2}(k_1+k_2)\, (\omega+\zeta) + \cosh (\tfrac{1}{2}(k_2-k_1)\, \omega + T)\}^2$$
$$\times \{e^{-\theta} \cosh \tfrac{1}{2}(k_1+k_2)\, (\zeta-\omega) + \cosh (\tfrac{1}{2}(k_2-k_1)\, \omega - T)\}^2. \tag{C 21}$$

From these definitions of h_1 and h_2 it is clear that

$$h_1(\omega, -\zeta, -T) = -h_1(\omega, \zeta, T) \tag{C 22}$$

and

$$h_2(\omega, -\zeta, -T) = +h_1(\omega, \zeta, T). \tag{C 23}$$

Hence under the change of variable $T^* = -T$ and $\zeta^* = -\zeta$, we obtain

$$I = -I. \tag{C 24}$$

Therefore I must equal zero, giving the required result.

Appendix D

In this section we establish the result (3.34). From the definitions of J, (3.28), and E, (3.12), we obtain the expression

$$\begin{aligned} k_2^4 J_{k_1} - k_1^4 J_{k_2} = \int_{-\infty}^{\infty} \Bigg[& \frac{E_1 E_2}{E^2} \{(k_1^4-k_2^4)\, x - \tfrac{3}{2} k_1^2 k_2^2 (k_1^2-k_2^2)\, t\} \\ & + \frac{2E_1 E_2}{E^3} \left\{ k_2^4 E_1 - k_1^4 E_2 - \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 (k_1^4-k_2^4)\, E_1 E_2 \right\} x - \frac{3E_1 E_2}{E^3} \\ & \times \left\{ k_2^2 E_1 - k_1^2 E_2 - \frac{(k_1-k_2)^2}{(k_1+k_2)^2} E_1 E_2 \right\} k_1^2 k_2^2\, t - \frac{8(k_1-k_2)}{(k_1+k_2)^3} (k_2^5+k_1^5) \frac{E_1^2 E_2^2}{E^3} \Bigg] \mathrm{d}x \\ = \int_{-\infty}^{\infty} \Bigg[& \frac{E_1 E_2}{E^2} \{(k_1^4-k_2^4)\, x - \tfrac{3}{2} k_1^2 k_2^2 (k_1^2-k_2^2)\, t\} \\ & + \frac{2E_1 E_2}{E^3} \left\{ \frac{k_1^5+k_2^5}{k_1+k_2} (E_1-E_2) + \frac{k_1^4-k_2^4}{k_1+k_2} E' \right\} x \\ & - \frac{3E_1 E_2}{E^3} \left\{ \frac{k_1^3+k_2^3}{k_1+k_2} (E_1-E_2) + \frac{k_1^2-k_2^2}{k_1+k_2} E' \right\} k_1^2 k_2^2\, t \\ & - \frac{8(k_1-k_2)}{(k_1+k_2)^3} (k_1^5+k_2^5) \frac{E_1^2 E_2^2}{E^3} \Bigg] \\ = \int_{-\infty}^{\infty} \Bigg[& \frac{2E_1 E_2}{E^3} \left\{ \frac{k_1^5+k_2^5}{k_1+k_2} x - \frac{3}{2} \frac{k_1^3+k_2^3}{k_1+k_2} k_1^2 k_2^2\, t \right\} (E_1-E_2) + \frac{E_1 E_2}{E^2} \left\{ \frac{k_1^4-k_2^4}{k_1+k_2} \right\} \\ & - \frac{8(k_1-k_2)}{(k_1+k_2)^3} (k_1^5+k_2^5) \frac{E_1^2 E_2^2}{E^3} \Bigg], \end{aligned} \tag{D 1}$$

the final expression being obtained by using an integration by parts. In the third expression in the integrand we write

$$\frac{(k_1-k_2)^2}{k_1+k_2}\frac{(E_1E_2)^2}{E^3} = \frac{E_1E_2}{E^3}(-E'-k_1E_1-k_2E_2), \tag{D 2}$$

so that

$$\int_{-\infty}^{\infty}\frac{(k_1-k_2)^2}{k_1+k_2}\frac{(E_1E_2)^2}{E^3} = \left[\frac{1}{2}\frac{E_1E_2}{E^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\tfrac{1}{2}(k_1+k_2)\frac{E_1E_2}{E^2}$$
$$-\tfrac{1}{2}(k_1+k_2)\int_{-\infty}^{\infty}\frac{E_1E_2(E_1+E_2)}{E^3}+\tfrac{1}{2}(k_1-k_2)\int_{-\infty}^{\infty}\frac{E_1E_2(E_2-E_1)}{E^3}. \tag{D 3}$$

The integral
$$\int_{-\infty}^{\infty}\frac{E_1E_2}{E^2}$$

appearing in (D 1) and (D 3) is the integral J, so that the contribution it gives to the right-hand side of (3.34) is zero. Hence from (3.34) we obtain

$$[k_1] = +\tfrac{8}{9}\epsilon^2\frac{(k_1-k_2)^2}{(k_1+k_2)^2}k_2^4\{I_1+I_2+I_3\}, \tag{D 4}$$

where

$$I_1 = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left(\frac{E_1E_2}{E^3}\left\{\frac{k_1^5+k_2^5}{k_1+k_2}x-\frac{3}{2}\frac{k_1^3+k_2^3}{k_1+k_2}k_1^2k_2^2t\right\}(E_2-E_1)\,\mathrm{d}x\right\}\frac{\partial J}{\partial t}\mathrm{d}t, \tag{D 5}$$

$$I_2 = -2\frac{k_1^5+k_2^5}{k_1^2-k_2^2}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{E_1E_2(E_1+E_2)}{E^3}\mathrm{d}x\right\}\frac{\partial J}{\partial t}\mathrm{d}t, \tag{D 6}$$

and

$$I_3 = +2\frac{k_1^5+k_2^5}{(k_1+k_2)^2}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\frac{E_1E_2(E_2-E_1)}{E^3}\mathrm{d}x\right\}\frac{\partial J}{\partial t}\mathrm{d}t. \tag{D 7}$$

Following the formulation of Appendix C we write

$$\frac{E_1E_2(E_2-E_1)}{E^3}$$
$$= -\frac{\sinh\left(\tfrac{1}{2}(k_2-k_1)x-\tfrac{1}{4}(k_2^3-k_1^3)t\right)}{4\{\mathrm{e}^{-\theta}\cosh\left(\tfrac{1}{2}(k_1+k_2)x-\tfrac{1}{4}(k_1^3+k_2^3)t+\theta\right)+\cosh\left(\tfrac{1}{2}(k_2-k_1)x-\tfrac{1}{4}(k_2^3-k_1^3)t\right)\}^3}\dagger \tag{D 8}$$

and, regarding I_1 as a triple integral as in (C 7), we make the following substitutions:

$$\left.\begin{aligned} y &= \tfrac{1}{2}(k_2-k_1)\{x-\tfrac{1}{2}(k_1^2-k_1k_2+k_2^2)t\}+\frac{(k_2-k_1)\theta}{k_2+k_1},\\ \eta &= \tfrac{1}{2}(k_2-k_1)\{\xi-\tfrac{1}{2}(k_1^2-k_1k_2+k_2^2)t\}+\frac{(k_2-k_1)\theta}{k_2+k_1},\\ \tau &= \tfrac{1}{2}k_1k_2(k_2-k_1)t+\frac{(k_2-k_1)\theta}{k_2+k_1}.\end{aligned}\right\} \tag{D 9}$$

† See note on page 479. Here we make the assumption $k_2 > k_1$.

Then I_1 becomes

$$I_1 = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2(k_2-k_1)^2}$$
$$\times\frac{\sinh(y-\tau)\sinh(\eta-\tau)\{Ay+B\tau-C\theta\}\,\mathrm{d}y\,\mathrm{d}\eta\,\mathrm{d}\tau}{\left\{\mathrm{e}^{-\theta}\cosh\frac{k_2+k_1}{k_2-k_1}y+\cosh(y-\tau)\right\}^3\left\{\mathrm{e}^{-\theta}\cosh\frac{k_2+k_1}{k_2-k_1}\eta+\cosh(\eta-\tau)\right\}^3} \tag{D 10}$$

where A, B and C are constants with

$$C=\frac{(k_1-k_2)^2}{k_1k_2(k_1+k_2)}\{(k_1^2+k_1k_2+k_2^2)^2-2k_1^2k_2^2\}. \tag{D 11}$$

Then under the transformation $t=-t^*, y=-y^*, \eta=-\eta^*$ we obtain

$$I_1=-I_1+\frac{C\theta}{30(k_2-k_1)^2}\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau \tag{D 12}$$

where
$$J_1(\tau)=\int_{-\infty}^{\infty}\frac{(30)^{\frac{1}{2}}\sinh(y-\tau)}{\{\beta\cosh(y/\beta)+\cosh(y-\tau)\}^3}\mathrm{d}y \tag{D 13}$$

and
$$\beta=\frac{k_2-k_1}{k_2+k_1}. \tag{D 14}$$

Therefore we have the result that

$$I_1=+\frac{C\theta}{60(k_2-k_1)^2}\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau$$
$$=+\frac{\{(k_1^2+k_1k_2+k_2^2)^2-2k_1^2k_2^2\}}{60k_1k_2(k_1+k_2)}\log\left(\frac{k_2+k_1}{k_2-k_1}\right)\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau. \tag{D 15}$$

Using the same set of transformations on I_2 the equation corresponding to (D 12) is

$$I_2=-I_2, \tag{D 16}$$

so that
$$I_2\equiv 0, \tag{D 17}$$

and with the definition of (D 13) I_3 is seen to be equal to

$$-\frac{k_1^5+k_2^5}{30(k_1+k_2)^2(k_1-k_2)^2}\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau. \tag{D 18}$$

The final expression for $[k_1]$ from (D 4) is then

$$[k_1]=-\frac{4\epsilon^2k_2^4}{270(k_1+k_2)^3}$$
$$\times\left\{\frac{2(k_1^5+k_2^5)}{(k_1+k_2)}-\frac{\{(k_1^3-k_2^3)^2-2k_1^2k_2^2(k_1-k_2)^2\}}{k_1k_2}\log\left(\frac{k_2+k_1}{k_2-k_1}\right)\right\}\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau$$

which in terms of β (see (D 14) is

$$[k_1]=-\frac{\epsilon^2k_2^5}{270(1+\beta)}\{1+10\beta^2+5\beta^4+2(7+10\beta^2-\beta^4)\beta^2\log\beta/(1-\beta^2)\}\int_{-\infty}^{\infty}J_1^2\,\mathrm{d}\tau. \tag{D 19}$$

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